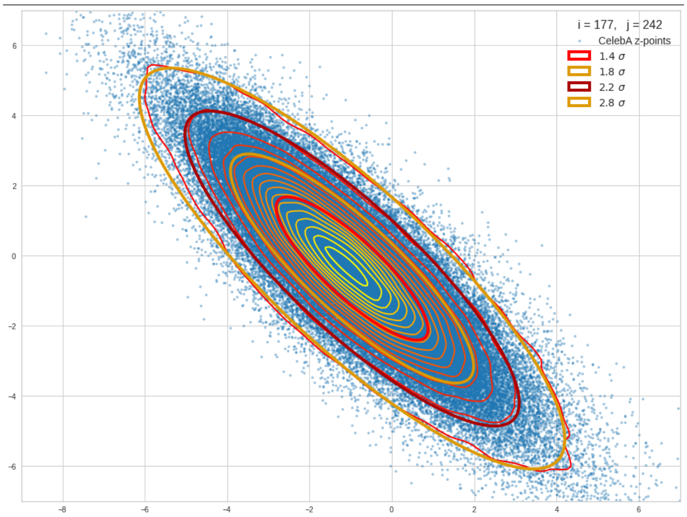
Ellipses and its quadratic form coefficients - getting the lengths of the semi-axes by evaluating extrema of the vector norm and by using a Lagrange multiplier



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Introduction and motivation: A “messy” way of deriving the lengths of semi-axes of an ellipse from its quadratic form matrix?

Many aspects of ellipses can be proven with rather different mathematical approaches - and not all of the possible approaches are equally efficient. Nevertheless, it is sometimes instructive to follow a certain complicated path from its start to its end. This is especially interesting when an “AI” as ChatGPT 5.x produces substantial errors along such a path and qualifies the whole endeavor as , I quote, “very messy”. Well, “messy” is a rather un-mathematical expression - and it indicates more a problem with limited capabilities of an AI than a lack of effectiveness. Sometimes, a cumbersome process consisting of many small, but elementary steps can be equally effective and sometimes even as efficient as a high end solution based on sophisticated knowledge and methods - and an AI should certainly be able to produce correct results with *any* chosen approach.

We all know very well that a “centered” ellipse can be represented by a quadratic form and as a level set of a respective function $F_E(x, y)$:

$$F_E(x,y) \,=\, \alpha\, x^2 \,+\, \beta\, x\, y \,+\, \gamma\, y^2 \,=\, C \,=\, const. \tag{1}$$

"Centered” means that the symmetry center of the ellipse where its main axes cross each other coincides with the origin of a Cartesian coordinate system [CCS]. The quadratic form describes the coordinates x and y of points on an ellipse or, if you like to think in terms of vectors, the components of respective position vectors from the CCS-origin to points on the ellipse.

The equation above can be rewritten with the help of a symmetric, invertible and even positive-definite matrix $\mathbf{Q}_\text{E}$ with elements α, β and γ

$$\mathbf{Q}_\text{E} \,=\, \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix} . \tag{2}$$

as a condition imposed on vectors $\mathbf{v}_e = (x, y)^\text{T}$ to points on the ellipse:

$$(\mathbf{v}_e)^\text{T}\, \mathbf{Q}_\text{E}\; \mathbf{v}_e \,=\, C\,. \tag{3}$$

The matrix elements must fulfill some inequalities:

$$\alpha \,\geq\, 0 \quad\wedge\quad \gamma \,\geq\, 0 \quad\wedge\quad \beta^2 \,<\, 4\,\alpha\,\gamma\,. \tag{4}$$

The determinant of $\mathbf{Q}_\text{E}$is given by

$$\det(\mathbf{Q}_\text{E}) \,=\, \frac{1}{4}\,\left(4\alpha\gamma - \beta^2\right) \,=\, \frac{1}{a^2\, b^2}\,, \tag{5}$$

with a and b being the lengths of the semi-axes[1]. These lengths are given by the *eigenvalues* of matrix $\mathbf{Q}_\mathsf{E}$, i.e. as solutions of the eigenvalue-problem

$$\left[\mathbf{Q}_\mathsf{E} - \lambda_{1/2} \, \mathbb{I}_2 \right] \circ \boldsymbol{\xi}_{1/2} = \mathbf{0}$$

for eigenvectors $\boldsymbol{\xi}_{1/2}$. "$\mathbb{I}_2$" is the (2x2) identity matrix. Solving the eigenvalue problem is an efficient way of determining the semi-axes' lengths and it is easy to realize in numerical procedures. The direction of the semi-axes is given by the eigenvectors $\boldsymbol{\xi}_{1/2}$. I just give you the solutions for the eigenvalues below:

$$\lambda_1 = \frac{1}{2} \left((\alpha + \gamma) - \left[\beta^2 + (\gamma - \alpha)^2 \right]^{1/2} \right) , \tag{6}$$

$$\lambda_2 = \frac{1}{2} \left((\alpha + \gamma) + \left[\beta^2 + (\gamma - \alpha)^2 \right]^{1/2} \right) . \tag{7}$$

Note that a and b relate to $\lambda_1$and λ_2 as reciprocate square roots of the eigenvalues:

$$a = \frac{1}{\sqrt{\lambda_1}} , \tag{8}$$

$$b = \frac{1}{\sqrt{\lambda_2}} . \tag{9}$$

Examples of valid eigenvectors are:

$$\lambda_1 : \quad \boldsymbol{\xi}_1 = \left(\frac{1}{\beta} \left((\alpha - \gamma) - \left[\beta^2 + (\gamma - \alpha)^2 \right]^{1/2} \right) , 1 \right)^\mathsf{T} , \tag{10}$$

$$\lambda_2 : \quad \boldsymbol{\xi}_2 = \left(\quad \left((\alpha - \gamma) + \left[\beta^2 + (\gamma - \alpha)^2 \right]^{1/2} \right) , \beta \right)^\mathsf{T} . \tag{11}$$

A "messy" approach?

However, one can choose a very different mathematical approach to find the lengths of the semi-axes. One can rely on calculus instead of using Linear Algebra: We can take one of the two graph functions of an ellipse, calculate x- and y-components and derive a respective function $r_e(x, y)$ for the standard l_2-norm (= length) of the vectors. Then we can differentiate this function $r_e(x, y)$ and set it to zero to find extrema. This is what ChatGPT 5.2 (free version) called "a very messy way" and strangely enough failed to get through it. Instead it invented erroneous shortcuts. As I am getting very suspicious when an algorithm uses expressions loaded with human feeling, I followed ChatGPT's way along the messy way - and unfortunately found not one, but two mistakes. GPT acknowledged its errors when asked to prove its steps in detail - and argued that it had just wanted

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[1] This is common knowledge, but you find proofs in ellipse related articles at the blog site "machine-learning.anracom.com"

to avoid complicated terms with nested square roots. So, never trust the answers of an AI without verifying it! In this case the errors were shockingly elementary.

This paper shows the path through the "messiness". And, of course, we will get the same result as via the efficient way guided by Linear Algebra.

Approach via Lagrange Multiplier

For the sake of completeness, I will also apply the powerful method of a Lagrange multiplier to derive the vectors of the ellipse which have maximum and minimum length. The advantage of this method is that we also get the direction of the semi-axes as a side product. Lagrange multipliers are typically applied when we look out for of extrema of a multivariate function under (multiple) constraint conditions. In our case there is just one constraint: The searched for extremal vectors must be members of the ellipse. The vectors we want to find are those which have minimum or maximum length. This gives us the extremum condition. One can show that the gradients of the function used to set a constraint and the gradient of the function whose extrema are requested must be parallel - and can therefore be linked by a multiplier λ.

The quadratic form function, its level sets and related graph functions

In this section, we first look at level sets of our quadratic form function $F_E(x, y)$. Then we exploit the assertion of the implicit function theorem of multivariate calculus that such level sets are locally equivalent to a graph. We identify the two graph functions $y_{U/L}(x) = G(x)$, which together cover an ellipse completely - each of them being defined on the same interval of $\mathbb{R}$ along the x-axis, but representing different parts of the ellipse's curve in the $\mathbb{R}^2$.

The quadratic form function, its contour ellipses and the related graph functions

Our quadratic form function $F_E(x,\ y)$ is a scalar function $F_E : \mathbb{R}^2 \to \mathbb{R}$. It fulfills the conditions for the implicit function theorem. This means that there exist parameterized functions which create graphs that piecewise cover level sets of F_E. Such level sets are defined by 1: $F_E(x,\ y) = C = const.$ In this section we list respective graph functions. The level sets in our case will of course be 1-dimensional contour lines (of $F_E(x,\ y)$) in the $\mathbb{R}^2$.

Let us define a set of level constants $\{C_i = const. \mid i = 1, ..k\}$. Then we get a corresponding number of different level sets of F_E, each composed of vectors of a 1-dimensional "surface" S_{Ci} in the $\mathbb{R}^2$ by

$$S_{Ci} \,=\, F_E^{-1}(C_i)\,, \ \text{ with } \ C_i = const. \ \wedge \ i = 1, ...k\,.$$

What kind of curves do we expect to be the outcome? We may intuitively think of ellipses. But how to describe them by ordinary functions? Fortunately, in our simple case the solutions for the graph functions are easy to find[2]: We just have to solve the eq. (1) for y:

$$F_E(x,y) \;=\; \alpha * x^2 \;+\; \beta * x\,y \;+\; \gamma * y^2 \;=\; C \tag{12}$$

$$\Rightarrow \quad y_{U/L}(x,C) \;=\; \frac{1}{2\,\gamma}\,\left(\pm\left[\,4\,\gamma\,C \;-\; \left(4\,\alpha\,\gamma - \beta^2\right)\,x^2\,\right]^{1/2} \;-\; \beta\,x\right)\,. \tag{13}$$

We find two functions $y_U(x)$ and $y_L(x)$. The range of x is limited:

$$4\,\gamma\,C \;-\; \left(4\,\alpha\,\gamma - \beta^2\right)\,x^2 \geq\; 0$$

$$\Rightarrow \quad x_{min} \;=\; -\sqrt{\frac{4\,\gamma\,C}{\left(4\,\alpha\,\gamma - \beta^2\right)}} \;\leq\; x \;\leq +\sqrt{\frac{4\,\gamma\,C}{\left(4\,\alpha\,\gamma - \beta^2\right)}} \;=\; x_{max}\,.$$

Remember that α, $\gamma > 0$. One could discuss whether to include the border points as the derivative there will get infinite. The respective level set or "surface" S_C is given as the union of two sets of vectors:

$$\Gamma_U \;=\; \left\{\,(x, y_U(x,C))^{\mathsf{T}} \mid x \in\; [x_{min},\, x_{max}[\;\right\}\,,$$

$$\Gamma_L \;=\; \left\{\,(x, y_L(x,C))^{\mathsf{T}} \mid x \in\;]x_{min},\, x_{max}]\,\right\}\,,$$

$$S_C \;=\; \Gamma_U \,\cup\, \Gamma_L\,.$$

In our case the "surface" is 1-dimensional and therefore just a (closed) curve. Let us plot the constituting curves Γ_U and Γ_L for $C = 1$ and different α, β and γ:

Figure 2 contains plots for other values of C:

We see clearly that covering a complete ellipse requires both functions. The ellipses for different values of C are all concentric.

The position of the maximum and minimum of an ellipses' graph functions

For the sake of completeness I add formulas for the positions of the points x_{str} with maximum and minimum values of $y_{U/L}(x,C)$. The derivative of the graph functions is given by

$$y'_{U/L}(x) \;=\; \frac{1}{2\,\gamma}\,\left[\mp\frac{\left(4\,\alpha\,\gamma - \beta^2\right)\,x}{\left[\,4\,\gamma\,C \;-\; \left(4\,\alpha\,\gamma - \beta^2\right)\,x^2\,\right]^{1/2}} \;-\; \beta\right] \tag{14}$$

and the points x_{xtr} for which $y'_{U/L}(x_{str}) = 0$ are given by the coordinates:

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[2] For general functions *F(x,y)* it can become difficult, if not impossible to determine graph functions of level sets.

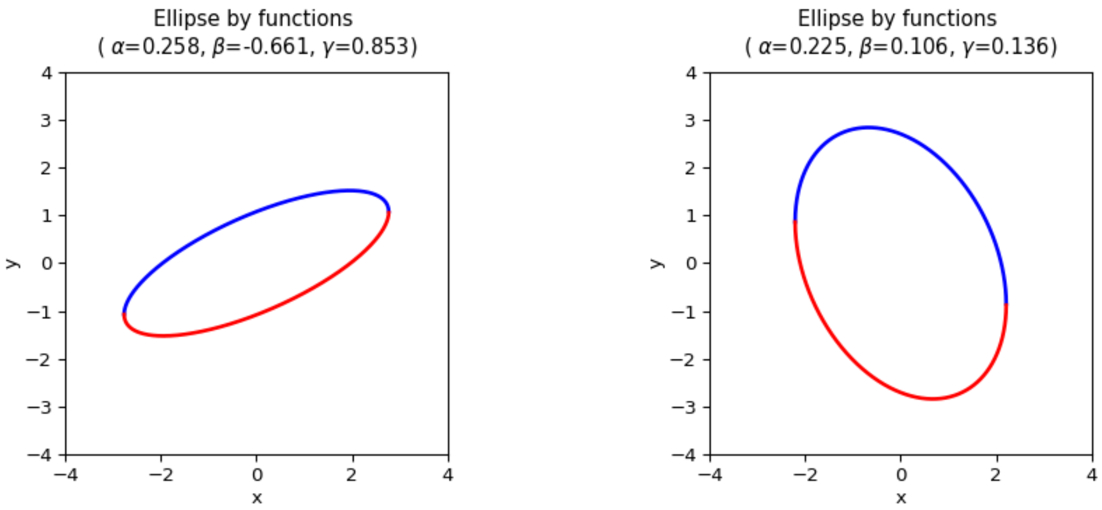

Figure 1: Plots for ellipses constructed by functions $y_u(x)$ in blue and $y_l(x)$ in red. The functions were based on the coefficients of the quadratic form function $F_E(x, y)$. The constant C was in both cases $C = 1$.

$$\begin{aligned} x_{xtr} &= \pm \beta \sqrt{C/\alpha}\, (4\,\alpha\gamma - \beta^2)^{-1/2}\,, \\ y_{xtr} &= \frac{1}{2\,\gamma} \left(\pm \left[4\,C\,\gamma \,-\, \left(4\,\alpha\gamma - \beta^2\right)\, x_{xtr}^2 \right]^{1/2} - \beta\, x_{xtr} \right) \end{aligned}$$

Actually, both formulas together tell you which choice to make to get a maximum and what to do to get a minimum. It depends on the last term on the right side of the expression for y_{str}. Remember that $\gamma > 0$ and that x was restricted to make the term under the square root positive. I leave it to the reader to filter the conditions. The following figures show you the position of the maxima and minima calculated for different ellipses and their values of C.

Derivation of the semi-axes' lengths by evaluating extrema of the vector lengths

After having found the graph functions of an ellipse we want get the lengths of the ellipses' semi-axes a and b via means of calculus. Actually, our task corresponds to just a different type of seeking extrema - this time for the length (or the squared length) of a position-vector from the CCS origin to the ellipse's curve. We start from

$$y_{U/L}(x, C) \,=\, \frac{1}{2\,\gamma} \left(-\beta\, x \pm \left[4\,\gamma\, C \,-\, \left(4\,\alpha\gamma - \beta^2\right)\, x^2 \right]^{1/2} \right) .$$

For our task we set $C = 1$. If later required, we can always correct for other values of C by dividing the coefficients α, β and γ by C. We focus on $y_U(x)$ and introduce the following terms

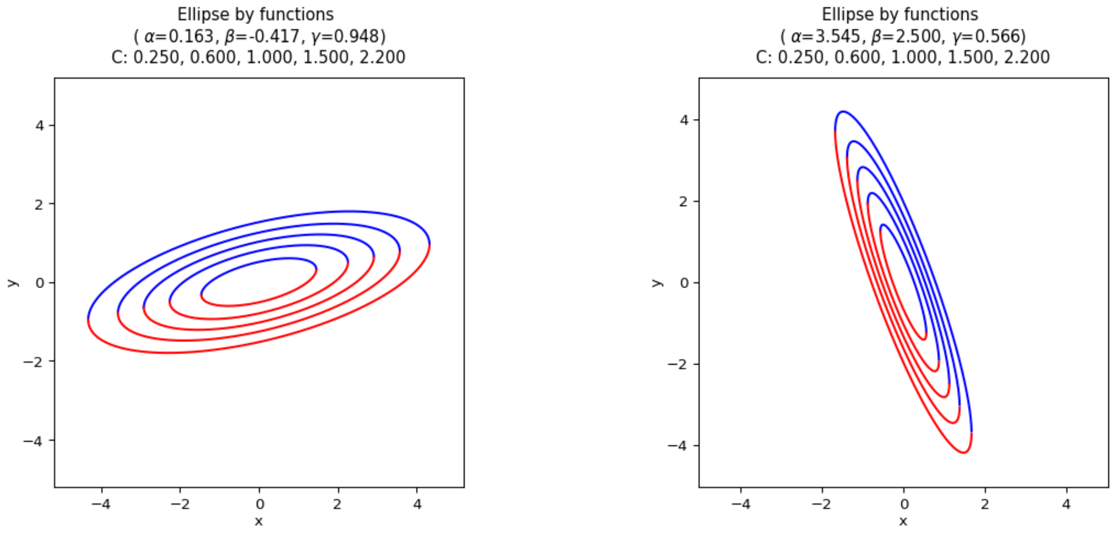

Figure 2: Plots for ellipses constructed by functions $y_U(x)$ in blue and $y_L(x)$ in red. The functions were based on selected values of coefficients of the quadratic form function $F_E(x, y)$. The constant C was in both cases varied $C = 0.25,\ 0.6,\ 1.0,\ 1.5,\ 2.2$.

and abbreviations

$$D \;=\; 4\,\alpha\,\gamma - \beta^2\,, \tag{15}$$

$$R(x) \;=\; \sqrt{4\,\gamma - D\,x^2}\,, \tag{16}$$

$$N \;=\; 4\,\gamma^2 \,+\, 2\,\beta^2 \,-\, 4\,\alpha\gamma\,, \tag{17}$$

$$S \;=\; \left[4\,\beta^2\,D \,+\, N^2\right]^{1/2}\,. \tag{18}$$

We need to get the squared length of a position vector to a point on the ellipse:

$$r^2(x) \;=\; x^2 + y^2(x)\,. \tag{19}$$

Using D,$R(x)$ and N, we find

$$y_U^2 \;=\; \frac{1}{4\,\gamma^2}\,\left[\beta^2\,x^2 \,-\, 2\,\beta\,R(x)\,x \,+\, 4\,\gamma \,-\, D\,x^2\right]\,, \tag{20}$$

$$\Rightarrow\quad y_U^2 \;=\; \frac{1}{4\,\gamma^2}\,\left[(2\,\beta^2 - 4\,\alpha\,\gamma)\,x^2 \,-\, 2\,\beta\,R(x)\,x \,+\, 4\,\gamma\,C\right]\,, \tag{21}$$

$$\Rightarrow\quad r^2 \;=\; x^2 + y_U^2 \;=\; \frac{1}{4\,\gamma^2}\,\left[N\,x^2 \,-\, 2\,\beta\,R(x)\,x \,+\, 4\,\gamma\,C\right]\,. \tag{22}$$

Now, we must find the extrema of $r(x)$. It is actually easier to take the derivative of $r^2(x)$, which, of course, will give us the same results. $R(x)$ with its square root nevertheless causes some trouble.

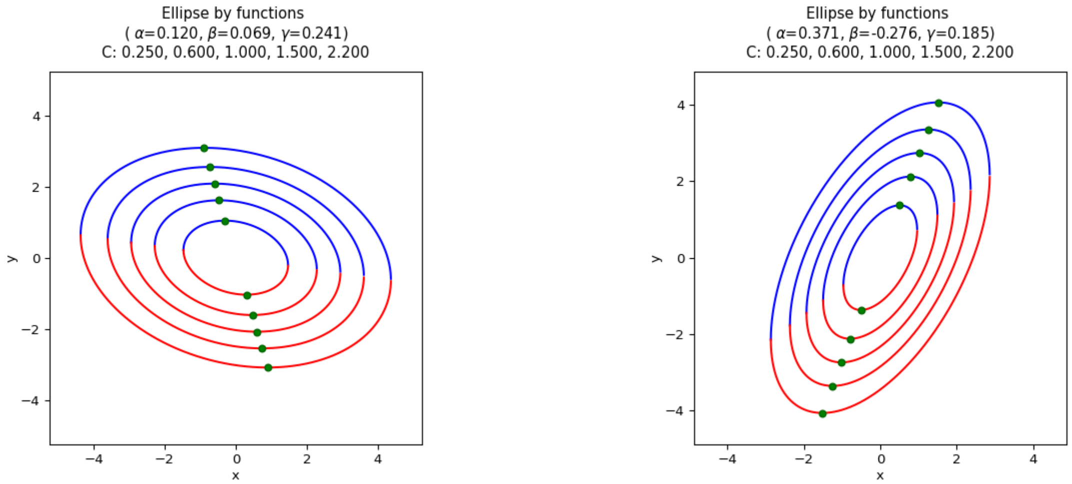

Figure 3: Extrema of various ellipses. The positions of the points of the maxima and minima were calculated by setting the derivatives of their graph functions to zero.

Computing the derivative and setting it to zero results in

$$\frac{d\, r^2(x)}{d\, x} \;=\; \frac{1}{4\,\gamma^2}\,\frac{1}{R(x)}\,\left[\,N\,x\,R(x)\;-\;\beta\,R^2(x)\;-\;\beta\,D\,x^2\,\right] \;=\; 0.$$

By expanding $R^2(x)$ and rearranging terms we get the central equation we need to solve

$$N\,x\,R(x)\;+\;\beta\,R^2(x)\;-\;\beta\,D\,x^2\;=\;0\,, \tag{23}$$

$$\Rightarrow \qquad N\,x\,\sqrt{4\,\gamma\,C - D\,x^2}\;=\;2\,\beta\,(\,D\,x^2 - 2\,\gamma)\,. \tag{24}$$

We get rid of the square root by squaring both sides:

$$N^2\,x^2\,(4\,\gamma\,C - D\,x^2)\;=\;4\,\beta^2\,(\,D\,x^2\;-\;2\,\gamma)^2\,. \tag{25}$$

Reordering and adding terms to complete the quadratic expression results in

$$\left(x^2 - 2\,\frac{\gamma}{D}\right)^2 \;=\; 4\,\frac{\gamma^2}{D^2}\;-\;16\,\frac{\beta^2\,\gamma^2}{D\,\left(4\,\beta^2\,D\;+\;N^2\right)}\,, \tag{26}$$

$$\Rightarrow \quad x^2 \;=\; 2\,\frac{\gamma}{D}\left[1\;\pm\;\left(1\;-\;\frac{4\,\beta^2\,D}{\left(4\,\beta^2\,D\;+\;N^2\right)}\right)^{1/2}\right]\,, \tag{27}$$

$$\Rightarrow \quad x^2 \;=\; 2\,\frac{\gamma}{D}\left[1\;\pm\;\left(1\;-\;\frac{N}{S}\right)\right]\,. \tag{28}$$

Which sign to follow? The eventual formula for $r^2(x)$ shows that we must choose the positive sign to get extreme values of r^2. A more detailed case analysis, which I omit in this paper, shows that it will always give you the major semi-axis a. We follow this branch.

$$x_a^2 \,=\, 2\,\frac{\gamma}{D\,S}\,(S\,+\,N)\;. \tag{29}$$

After having taken the square root, there is yet another case decision to be made:

$$x_a \,=\, \mp\,\left[2\,\frac{\gamma}{D\,S}\,(S\,+\,N)\right]^{1/2}\;. \tag{30}$$

We will see that the decision depends on the discussion of a specific term. But let us first proceed with $r_a^2 \,=\, y^2(x_a) + x_a^2$:

$$r_a^2 \,=\, \frac{1}{4\,\gamma^2}\,\left[\left(\beta^2 - D + 4\,\gamma^2\right)\,x_a^2 \,+\, 4\,\gamma \,\pm\, 2\,\beta\,R(x_a)\,*\,x_a\right]\;. \tag{31}$$

I left the sign of the last term open for a moment because x may change sign and it is unclear, yet, whether it would do this in a coordinated way with β and $R(x)$. So, it makes sense to investigate the rightmost term a bit closer. The first thing, we learn is

$$\begin{aligned}\left[4\,\gamma \,-\, D\,x_a^2\right]^{1/2} \,&=\, \left[4\,\gamma \,-\, \frac{2\,D\,\gamma}{D\,S}\,(S+N)\right]^{1/2} &(32)\\ &=\, \frac{1}{(DS)^{1/2}}\,(2\,\gamma\,D)^{1/2}\,[S-N]^{1/2}\;. &(33)\end{aligned}$$

Therefore, the last term is equal to

$$\begin{aligned}2\,\beta\,R(x_a)\,*\,x_a \,&=\, 2\,\beta * \frac{1}{(DS)^{1/2}}\,(2\,\gamma\,D)^{1/2}\,[S-N]^{1/2} * \frac{1}{(DS)^{1/2}}\,(2\,\gamma)^{1/2}\,[S+N]^{1/2} &(34)\\ &=\, \frac{2\,\gamma}{DS} * 4\,\beta^2 D\,. &(35)\end{aligned}$$

The nice and helpful thing here is the product $(S+N)*(S-N) \,=\, S^2 - N^2$:

$$(S+N)*(S-N) \,=\, S^2 - N^2 \,=\, 4\,\beta^2\,D\,, \tag{36}$$

As both S,D and β^2 are always positive, it becomes clear that an extremum of r_a^2 requires the positive sign of the last term. Now, expanding x_a^2 in the first term and extracting $2\gamma/(SD)$ to the front of the bracket we reach the intermediate result

$$
\begin{aligned}
r_a^2 \;&=\; \frac{1}{4\gamma^2}\,\frac{2\gamma}{DS} * \left[\beta^2 - D + 4\gamma^2)\,(S+N) \,+\, 2\,D\,S \,+\, 4\,\beta^2\,D\right] && (37)\\
&=\; \frac{1}{2\gamma}\,\frac{1}{DS} * \left[\beta^2 S \,+\, DS \,+\, 4\gamma^2 S \,+\, \beta^2 N \,-\, DN \,+\, 4\gamma^2 N \,+\, 4\,\beta^2 D\right] && (38)\\
&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} \,+\, \frac{1}{2\gamma} \,+\, \frac{1}{2\gamma}\,\frac{1}{DS} * \left[4\gamma^2 S \,+\, \beta^2 N \,-\, DN \,+\, 4\gamma^2 N \,+\, 4\,\beta^2 D\right]\,. && (39)
\end{aligned}
$$

There is no way around of expanding all the terms on the right side step by step:

$$
\begin{aligned}
r_a^2 \;&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} + \frac{1}{2\gamma} + \frac{1}{2\gamma}\,\frac{1}{DS} * \left[4\gamma^2 S \,+\, 4\,\gamma^2\,\beta^2 \,+\, (\beta^2 + D)^2 \,-4\,\gamma^2\,D + 4\gamma^2 N\right] && (40)\\
&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} + \frac{1}{2\gamma} + \frac{2\gamma}{D} + \frac{1}{2\gamma}\,\frac{1}{DS} * \left[4\,\gamma^2\,\beta^2 \,+\, 16\,\alpha^2\,\gamma^2 \,-4\,\gamma^2\,D + 4\gamma^2 N\right] && (41)\\
&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} + \frac{1}{2\gamma} + \frac{2\gamma}{D} + \frac{1}{2\gamma}\,\frac{1}{DS} * \left[8\,\gamma^2\,\beta^2 \,+\, 16\,\alpha^2\,\gamma^2 \,-16\,\alpha\,\gamma^3 + 4\gamma^2 N\right] && (42)\\
&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} + \frac{1}{2\gamma} + \frac{2\gamma}{D} + \frac{8\gamma}{DS} * \left[\beta^2 \,+\alpha^2 \,-2\,\alpha\,\gamma + \gamma^2\,\right] && (43)\\
&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} + \frac{1}{2\gamma} + \frac{2\gamma}{D} + \frac{8\gamma}{DS} * \left[\beta^2 \,+\, (\gamma-\alpha)^2\right]\,. && (44)
\end{aligned}
$$

We are almost there. We just have to evaluate S :

$$
\begin{aligned}
S \;&=\; \left[4\,\beta^2\,(4\,\alpha\gamma - \beta^2) \,+\, \left(4\gamma^2 + 2\beta^2 - 4\,\alpha\gamma\right)^2\right]^{1/2} && (45)\\
&=\; \left[16\,\gamma^2\,\left[\beta^2 \,+\, (\gamma-\alpha)^2\right]\right]^{1/2} && (46)\\
\Rightarrow \qquad S \;&=\; 4\,\gamma\,\left[\beta^2 \,+\, (\gamma-\alpha)^2\right]^{1/2}\,. && (47)
\end{aligned}
$$

The appearance of a similar term as in the expression of r_a^2 encourages us to go the final steps for determining r_a^2:

$$
\begin{aligned}
r_a^2 \;&=\; \frac{1}{2\gamma}\,\frac{\beta^2}{D} + \frac{1}{2\gamma} + \frac{2\gamma}{D} + \frac{2}{D} * \left[\beta^2 \,+\, (\gamma-\alpha)^2\right]^{1/2} && (48)\\
&=\; \frac{1}{2\,\gamma\,D} * \left[\beta^2 \,+\, D \,+\, 4\gamma^2 \,+4\gamma\,\left[\beta^2 \,+\, (\gamma-\alpha)^2\right]^{1/2}\right] && (49)\\
&=\; \frac{1}{2\,\gamma\,D} * \left[\beta^2 \,+\, (4\alpha\gamma - \beta^2) \,+\, 4\gamma^2 \,+4\gamma\,\left[\beta^2 \,+\, (\gamma-\alpha)^2\right]^{1/2}\right] && (50)\\
&=\; \frac{2}{D} * \left[\alpha \,+\gamma \,+\, \left(\beta^2 \,+\, (\gamma-\alpha)^2\right)^{1/2}\right]\,. && (51)
\end{aligned}
$$

In an eventual step we can show that this is equal to the solution of the eigenvalue problem:

$$r_a^2 = \frac{2}{(\alpha+\gamma) - \left[\beta^2 + (\gamma-\alpha)^2\right]^{1/2}} \,. \tag{52}$$

The proof is simple. We have to verify the following identity:

$$\frac{2}{D} * \left[(\alpha+\gamma) + \left(\beta^2 + (\gamma-\alpha)^2\right)^{1/2}\right] = \frac{2}{(\alpha+\gamma) - \left[\beta^2 + (\gamma-\alpha)^2\right]^{1/2}} \,. \tag{53}$$

This means

$$\left[(\alpha+\gamma) + \left(\beta^2 + (\gamma-\alpha)^2\right)^{1/2}\right] * \left[(\alpha+\gamma) - \left(\beta^2 + (\gamma-\alpha)^2\right)^{1/2}\right] = D \,. \tag{54}$$

An expansion of the product on the left side and canceling terms leads to the claim

$$4\,\alpha\,\gamma - \beta^2 = D \,,$$

which indeed is true.

An analogous analysis for the square of the minor semi-axis r_b^2 gives us a similar result - with only a difference in a sign.

We can summarize:

- The x-coordinate of the end-points of the semi-axes are given by

$$\text{major axis:} \quad \beta > 0: \quad x_a = -\left[2\,\frac{\gamma}{D\,S}\,(S+N)\right]^{1/2} . \tag{55}$$

$$\beta < 0: \quad x_a = +\left[2\,\frac{\gamma}{D\,S}\,(S+N)\right]^{1/2} , \tag{56}$$

$$\text{minor axis:} \quad \beta > 0: \quad x_b = +\left[2\,\frac{\gamma}{D\,S}\,(S-N)\right]^{1/2} , \tag{57}$$

$$\beta < 0: \quad x_b = -\left[2\,\frac{\gamma}{D\,S}\,(S-N)\right]^{1/2} , \tag{58}$$

- The lengths of the semi-axes are:

$$\text{major axis:} \quad a^2 = r_a^2 = \frac{2}{(\alpha+\gamma) - \left[\beta^2 + (\gamma-\alpha)^2\right]^{1/2}} \,, \tag{59}$$

$$\text{minor axis:} \quad b^2 = r_b^2 = \frac{2}{(\alpha+\gamma) + \left[\beta^2 + (\gamma-\alpha)^2\right]^{1/2}} \,. \tag{60}$$

From the structure of the formulas it is clear that $x_a$and r_a refer to the major axis. The signs of the x-coordinates result from the fact that $y_U(x)$ for certain parameters only covers a certain region of the ellipse's curve. Therefore, the end-point of a semi-axis may have a negative x-coordinate.

Yet another approach: Tangential circles, a Lagrange multiplier and the angle and the lengths of the semi-axes

After having done something which ChatGPT 5.x had problems with, we dare to attack the problem of getting geometric properties of an ellipse by combining two different points of view:

- **Geometry:** Imagine circles around the origin of the CCS passing through points on the ellipse. The figures below show that we get special conditions for circles passing through the end points of the semi-axes. These special circles touch the ellipse tangentially - for the minor semi-axis from the inside of the convex ellipse and for the major semi-axis from the outside. For both curves we can say: At the extrema the vector lengths only change infinitesimally for a variation of the vector components. This means that both of the gradients of the respective functions must be perpendicular to the tangent(s) at these points, i.e. the gradient vectors must be parallel. This gives us a valuable condition, which we can directly exploit.
- **Lagrange Multiplier:** We are looking for extrema of a scalar function of two variables, namely the lengths of circle vectors, which at the same time must fulfill a **constraint condition**, namely being vectors of an ellipse. In such a situation we can introduce a Lagrange multiplier linking the gradients of both functions.

  $$\nabla\left(r^2(x)\right) \;=\; \lambda\,\nabla F_E(x). \tag{61}$$

  This will lead to a key equation helping us to find the extrema.

As both the the Lagrange approach and geometry lead to the same kind of condition, we follow the Lagrange multiplier:

$$\nabla r^2(x) \;=\; \begin{pmatrix} 2x \\ 2y \end{pmatrix} \;=\; \lambda\,\begin{pmatrix} 2\alpha x + \beta y \\ 2\gamma y + \beta x \end{pmatrix} \;=\; \lambda\,\nabla F_E(x)\,. \tag{62}$$

This results in two equations. We use the first one for λ, insert the result into the 2nd equation and simplify:

$$\lambda \;=\; \frac{2\,x}{2\alpha x + \beta y} \tag{63}$$

$$\Rightarrow \quad 0 = \beta\,(x^2 - y^2) \,+\, 2xy\,(\gamma - \alpha)\,. \tag{64}$$

This is a key equation which can be used together with the constraint of an ellipse to get information about the semi-axes. This equation strongly couples y and x. For a reasonable standard ellipse with semi-axes $a, b > 0$, we definitely know that the $x_{a/b}^2 > 0$. So, we can divide the equation

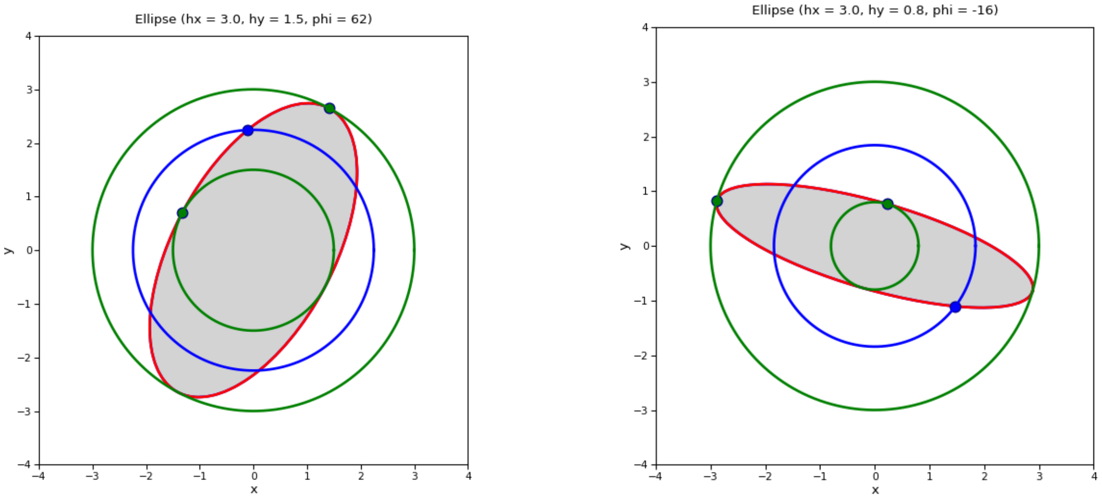

Figure 4: The plots show central circles through different points on an ellipse. Circles with extreme radii pass through points where the gradients of $r(x, y)$and $F_E(x, y)$ get parallel. Note that the blue circles doe not fulfill the condition of the Lagrange multiplier: The gradients are not aligned. The green circles passing the endpoints of the semi-axes, however, do fulfill the conditions imposed by the Lagrange multiplier: The gradients of both the ellipse's quadratic form function as well as of the function for the radius are parallel to each other and perpendicular to the tangent of the ellipse at the end points of the semi-axes.

above by x^2 and thus also y by x. We rewrite the equation in the following form:

$$\beta \left[1 - \left(\frac{y}{x}\right)^2\right] + 2 \left(\frac{y}{x}\right) (\gamma - \alpha) = 0 .$$

The fraction y/x has a geometrical meaning: It gives us the **tangent of the angle** which the position vector of a point forms with the x-axis of the CCS. Let us call it "t". So, we have a quadratic equation for t,

$$\beta (1 - t^2) + 2\, t\, (\gamma - \alpha) = 0 , \tag{65}$$

$$\Rightarrow \quad \beta\, t^2 - 2\, (\gamma - \alpha)\, t - \beta = 0 , \tag{66}$$

which we can solve

$$t = \frac{1}{\beta} \left[(\gamma - \alpha) \mp \sqrt{(\gamma - \alpha)^2 + \beta^2} \right] . \tag{67}$$

Due to the fact that our key equation was derived for the semi-axes (corresponding to vectors of extremal lengths), we conclude that this formula actually gives us the rotation angle of the ellipse (up to the periodicity of the tangent function). Here we would have to discuss the assignment of the angle to which of the semi-axes in detail, in particular in dependence of the signs of β and $\gamma - \alpha$. But we leave this discussion for another paper.

$$\tan(\phi) = \frac{1}{\beta} \left[(\gamma - \alpha) \pm \left[(\gamma - \alpha)^2 + \beta^2 \right]^{1/2} \right]$$

With the help of trigonometric identities we can bring the upper formula into another form by switching the sign of the argument

$$\cot(\phi) \;=\; -\tan(\phi) \;=\; \tan\left(-\phi\right) \;=\; \frac{1}{\beta}\left[\,(\alpha-\gamma) \;\mp\; \left[(\gamma-\alpha)^2+\beta^2\right]^{1/2}\right] .$$

Thus we have on the fly harvested a formula for the rotation angle of our ellipse dependent on the quadratic form coefficients α, β and γ :

$$\boxed{\tan(\phi_{1/2}) \;=\; \frac{\beta}{(\alpha-\gamma) \;\mp\; \left[(\gamma-\alpha)^2+\beta^2\right]^{1/2}} \,.} \tag{68}$$

Let us focus on the lengths of the semi-axes again. We insert t into the ellipse equation, which we have not used so far:

$$1 \;=\; \alpha x^2 + \beta\, x(tx) + \gamma t^2 x^2 \;, \tag{69}$$

$$\Rightarrow \quad x^2 \;=\; \frac{1}{\alpha+\beta\, t+\gamma\, t^2} \tag{70}$$

We do the same with the circle equation

$$r^2 \;=\; x^2(1+t^2) \;=\; \frac{1+t^2}{\alpha+\beta\, t+\gamma\, t^2} \,. \tag{71}$$

We introduce

$$S_R \;=\; \left[(\gamma-\alpha)^2+\beta^2\right]^{1/2} \tag{72}$$

and evaluate the numerator and the denominator of the expression for r^2

$$1+t^2 \;=\; 1+\frac{1}{\beta^2}\,\left[(\gamma-\alpha)^2 \pm 2\,(\gamma-\alpha)\,S_R\, + (\gamma-\alpha)^2+\beta^2\right] \tag{73}$$

$$= \frac{2}{\beta^2}\,\left[\beta^2+(\gamma-\alpha)^2 \pm S_R\right] \;=\; \frac{2}{\beta^2}\,\left[S_R^2\, \pm\, (\gamma-\alpha)\,S_R\right] \tag{74}$$

$$= \frac{2}{\beta^2}\,S_R\,\left[S_R\, \pm\, (\gamma-\alpha)\right] . \tag{75}$$

The terms in the denominator of the circle equation are

$$\alpha \,+\, \beta\, t \;=\; \alpha \,+\, (\gamma-\alpha) \,\pm\, S_R \;=\; \gamma \pm S_R \,, \tag{76}$$

$$\gamma\, t^2 \;=\; \frac{\gamma}{\beta^2}\,\left[(\gamma-\alpha)^2 \,\pm\, 2\,(\gamma-\alpha)\,S_R \,+\, \beta^2+(\gamma-\alpha)^2\right] . \tag{77}$$

Now comes a little, but very helpful trick: We add zero in the form $\beta^2 - \beta^2$ to get

$$\gamma\, t^2 \;=\; \frac{\gamma}{\beta^2}\;\left[2\beta^2 \;+ 2\,(\gamma-\alpha)^2 \;\pm\; 2\,(\gamma-\alpha)\,S_R \;-\; \beta^2\right] \tag{78}$$

$$\;=\; \frac{2\,\gamma}{\beta^2}\,S_R\;[S_R \pm (\gamma-\alpha)] \;-\; \gamma\,. \tag{79}$$

This means

$$\alpha + \beta\, t + \gamma\, t^2 \;=\; S_R\;\left[1 \;+\; \frac{2\,\gamma}{\beta^2}\;[S_R \pm (\gamma-\alpha)]\right]\;. \tag{80}$$

The reciprocate of r^2 becomes

$$\frac{\gamma}{r^2} \;=\; \frac{S_R\;\left[1 \;+\; \dfrac{2\gamma}{\beta^2}\;[S_R \pm (\gamma-\alpha)]\right]}{\dfrac{2}{\beta^2}\,S_R\;[S_R\;\pm\;(\gamma-\alpha)]} \tag{81}$$

$$=\; \frac{\beta^2}{2}\;\frac{1 \;+\; \dfrac{2\gamma}{\beta^2}\;[S_R \pm (\gamma-\alpha)]}{[S_R\;\pm\;(\gamma-\alpha)]}\;. \tag{82}$$

We follow the case for the "+"-sign. The negative case can later be done analogously. Another trick: To get rid of the square root S_R we multiply numerator and denominator with $(S_R - (\gamma-\alpha))$ and use

$$[S_R\;+\;(\gamma-\alpha)] * [S_R\;-\;(\gamma-\alpha)] \;=\; \beta^2 + (\gamma-\alpha)^2 - (\gamma-\alpha)^2 \;=\; \beta^2\,, \tag{83}$$

to find

$$\frac{1}{r^2} \;=\; \frac{\beta^2}{2}\;\frac{\left[1 \;+\; \dfrac{2\gamma}{\beta^2}\;[S_R + (\gamma-\alpha)]\right]\;[S_R\;-\;(\gamma-\alpha)]}{[S_R\;+\;(\gamma-\alpha)]\;[S_R\;-\;(\gamma-\alpha)]} \tag{84}$$

$$=\; \frac{1}{2}\;\left[S_R\;-\;(\gamma-\alpha)\;+\;2\,\gamma\,\beta^2/\beta^2\right] \tag{85}$$

$$=\; \frac{1}{2}\;[\,(\alpha\;+\gamma) +\;S_R] \;=\; \frac{1}{2}\;\left[(\alpha\;+\gamma) + \left[(\gamma-\alpha)^2 + \beta^2\right]^{1/2}\,\right]. \tag{86}$$

The case with the minus sign runs analogously. We come to the eventual formula for the *squared* lengths of the semi-axes and use our old letters $h_{1,}h_2$:

$$\boxed{\left(r_{a/b}\right)^2 \;=\; \frac{2}{(\alpha\;+\gamma) \mp \left[\,\beta^2\;+\;(\gamma-\alpha)^2\,\right]^{1/2}}}\;.$$

This is the same result as derived before. It shows the power of the method of Lagrange multipliers.

Conclusion

In this paper we have shown that a straightforward analysis of the calculus condition for extremal vector lengths allows us to compute the lengths of the semi-axes of an ellipse. The "messy" derivation, as ChatGPT 5.x (free version) qualified it, is at its core a straightforward one and, of course, gives us the same result as the eigenvalues of the quadratic form matrix. But we have also seen that applying the method of a Lagrange Multiplier has its advantages. We get some information on the orientation of the semi-axes on the fly.